Effects of viscous dissipation in natural convection

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The effect of viscous dissipation in natural convection is appreciable when the induced kinetic energy becomes appreciable compared to the amount of heat transferred. This occurs when either the equivalent body force is large or when the convection region is extensive. Viscous dissipation is considered here for vertical surfaces subject to both isothermal and uniform-flux surface conditions. A perturbation method is used and the first temperature perturbation function is calculated for Prandtl numbers from 10^{-2} to 10^4. The magnitude of the effect depends upon a dissipation number, which is not expressible in terms of the Grashof or the Prandtl number.

Introduction

It has been recognized that significant viscous dissipation may occur in natural convection in various devices which are subject to large decelerations or which operate at high rotative speeds. In addition, important viscous dissipation effects may also be present in stronger gravitational fields and in processes wherein the scale of the process is very large, e.g. on larger planets, in large masses of gas in space, and in geological processes in fluids internal to various bodies.

Such processes have received some attention particularly with respect to processes in rotating cavities. Investigation of the possibility of turbine blade cooling by natural circulation of an internal coolant resulted in an analysis by Lighthill (1953) which delineated flow patterns and predicted heat-transfer characteristics. Ostrach (1957) considers the effects of viscous dissipation in such passages with fully developed velocity and temperature profiles (i.e. a one-dimensional problem). These studies are primarily concerned with laminar processes in steady flow.

The various studies, and a general similarity analysis, show that the relative magnitude of the viscous dissipation effect is given by a dissipation number. This number is a completely independent parameter. It has no correspondence with the Prandtl number nor with the Grashof number, the latter being generally thought to be the important parameter for laminar instability and flow transition. Therefore, neither the value of the Prandtl number nor the upper limit for laminar processes preclude important viscous dissipation effects, as has been suggested.

The present paper is a two-dimensional boundary-layer analysis of the effects of viscous dissipation for external flow about a surface whose generator is parallel to the body force field which causes the motion. Prandtl numbers of 10^{-2}, 0·72, 10^2 and 10^4 are considered. A perturbation method is used. Circumstances of important viscous dissipation are discussed.

Analysis

The boundary-layer equations which apply for natural convection from a surface to a fluid at rest have been established by Schmidt and Beckmann through experimental observations of flow fields and by, for example, Ostrach (1953), through an order-of-magnitude analysis. As for forced convection, these equations need include neither an allowance for forces normal to the surface nor second derivatives parallel to the surface for the thin velocity and thermal layers which result when the relevant flow parameter is large. This parameter is the Grashof number for natural convection. Therefore, the boundary-layer equations for steady-state natural convection in a fluid of uniform, constant properties on a semi-infinite plate (parallel to the body force) are:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2}\pm g\rho\beta\theta, \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \tag{2}$$

and

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y} = \alpha\frac{\partial^2 \theta}{\partial y^2}+\frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2, \tag{3}$$

where u and v are the velocity components, θ is the temperature excess $(t-t_\infty)$, x is measured from the leading edge for the induced flow, and y is the distance out perpendicular to the plate. The plus and minus signs in (1) apply for heating and for cooling the fluid, respectively. The convection-induced pressure gradient neglected in (1) is considered in the Discussion. Boundary conditions which apply for all cases are:

$$\text{at } y = 0 \text{ for } x \geqslant 0, \quad u = v = 0; \qquad \text{as } y \to \infty, \quad u \to 0, \quad \theta \to 0.$$

There are a number of surface-temperature or heat-flux boundary conditions which are of interest. However, the uniform temperature and the uniform heat flux (q'') condition represent two different cases of frequent practical importance. These cases are analysed here. The additional boundary conditions are, therefore,

$$\text{at } y = 0: \quad \theta = (t_0-t_\infty), \quad \text{isothermal surface;}$$
$$q'' = -k\,\partial\theta/\partial y, \quad \text{uniform flux surface.}$$

The temperature excess θ is replaced by a generalized temperature ϕ defined for the two cases as follows:

$$\phi = \frac{\theta}{\theta_0} = \frac{(t-t_\infty)}{(t_0-t_\infty)}, \quad \text{isothermal;} \tag{4}$$

$$\phi = \frac{(t-t_\infty)}{(q''x/k)}\left(\frac{Gr_x}{5}\right)^{\frac{1}{5}}, \quad \text{uniform flux;} \tag{5}$$

where t_0 is the surface temperature for the isothermal case and Gr_x is the local Grashof number for the uniform flux case and is defined in equation (11). The surface boundary conditions in terms of ϕ are:

$$\text{at } y = 0: \ \phi = 1, \quad \text{isothermal surface;}$$
$$\phi' = -1, \quad \text{uniform flux surface.}$$

Introduction of a stream function $\psi(x,y)$ reduces equations (1), (2) and (3) to

$$\psi_y \psi_{yx} - \psi_x \psi_{yy} = \nu \psi_{yyy} \pm g\rho\theta \tag{6}$$

and

$$\psi_y \theta_x - \psi_x \theta_y = \alpha\theta_{yy} + (\nu/c_p)(\psi_{yy})^2. \tag{7}$$

The stream function ψ and temperature function ϕ are written in terms of perturbation functions f_i and ϕ_i as follows:

$$\psi = \nu m^{(m-1)/m} (Gr_x)^{1/m} [f_0 \pm m\epsilon f_1 \pm (m\epsilon)^2 f_2 \pm \ldots], \tag{8}$$

$$\phi = \phi_0 \pm m\epsilon\phi_1 \pm (m\epsilon)^2 \phi_2 \pm \ldots, \tag{9}$$

where $m = 4$ for the isothermal and 5 for the uniform-flux surface condition and the plus and minus signs apply for heating and cooling the fluid, respectively. The dimensionless quantity Gr_x is given as follows for the two cases:

$$Gr_x = |g\beta x^3(t_0 - t_\infty)/\nu^2|, \quad \text{isothermal}; \tag{10}$$

$$Gr_x = |g\beta x^4 q''/k\nu^2|, \quad \text{uniform flux}. \tag{11}$$

The functions $f_0, f_1, \ldots$ and $\phi_0, \phi_1, \ldots$ depend only upon the similarity variable

$$\eta = \frac{y}{x}\left(\frac{Gr_x}{m}\right)^{1/m}. \tag{12}$$

Comparison of (7) and (9) indicates that the second approximation function ϕ_1 will include the effect of viscous dissipation if ϵ is chosen as

$$\epsilon = g\beta x/c_p. \tag{13}$$

Thus ϵ is the local dissipation number, which is equal to the kinetic energy of the flow divided by the heat transferred to the fluid.

The result of choosing ϵ as in (13) is that f_0 and ϕ_0 are merely the well-known zero dissipation solutions given by Pohlhausen (1930), Schuh (1948), and Ostrach (1953) for the isothermal case and by Sparrow & Gregg (1956) for the uniform-flux case.

The differential equations (6) and (7) are written in terms of the perturbation functions given in (8) and (9). For each case, i.e. isothermal and uniform flux, four equations are obtained for the four functions f_0, ϕ_0, f_1 and ϕ_1. The functions f_2 and ϕ_2 do not include directly an effect of viscous dissipation since the dissipation term in the energy equation does not contain terms of even order in ϵ. Higher approximations are not considered further. The equations with the relevant boundary conditions are written below for the two cases.

Isothermal

$$f_0''' + 3f_0 f_0'' - 2f_0'^2 + \phi_0 = 0, \tag{14}$$

$$\phi_0'' + 3Prf_0\phi_0' = 0, \tag{15}$$

$$f_1''' + 7f_0''f_1 + 3f_0 f_1'' - 8f_0'f_1' + \phi_1 = 0, \tag{16}$$

$$\phi_1'' + Pr(7f_1\phi_0' + 3f_0\phi_1' - 4f_0'\phi_1 + f_0''^2) = 0; \tag{17}$$

at $\eta = 0, f_0 = f_0' = f_1 = f_1' = \phi_1 = 0, \phi_0 = 1$; as $\eta \to \infty, f_0', \phi_0, f_1'$ and $\phi_1 \to 0$.

Uniform flux

$$f_0''' + 4f_0 f_0'' - 3f_0'^2 + \phi_0 = 0, \tag{18}$$

$$\phi_0'' + Pr(4f_0\phi_0' - f_0'\phi_0) = 0, \tag{19}$$

$$f_1''' + 4f_0 f_1'' - 11 f_0' f_1' + 9 f_0'' f_1 + \phi_1 = 0, \tag{20}$$

$$\phi_1'' + Pr(9f_1\phi_0' + 4f_0\phi_1' - 6f_0'\phi_1 - f_1'\phi_0 + f_0''^2) = 0; \tag{21}$$

at $\eta = 0, f_0 = f_0' = f_1 = f_1' = \phi_1' = 0, \phi_0' = -1$; as $\eta \to \infty, f_0', \phi_0, f_1'$, and $\phi_1 \to 0$.

If the temperature effect of viscous dissipation is assumed to have a negligible effect (through buoyancy) on the velocity distribution (this amounts essentially to assuming $f_1 = 0$), the following simpler equations yield the second approximation ϕ_1 of the temperature distribution.

Isothermal

$$\phi_1'' + Pr(3f_0\phi_0' - 4f_0'\phi_1 + f_0''^2) = 0; \tag{22}$$

at $\eta = 0$, $\phi_1 = 0$; as $\eta \to \infty$, $\phi_1 \to 0$.

Uniform flux

$$\phi_1'' + Pr(4f_0\phi_1' - 6f_0'\phi_1 + f_0''^2) = 0; \tag{23}$$

at $\eta = 0$, $\phi_1' = 0$; as $\eta \to \infty$, $\phi_1 \to 0$.

Equation (22), or (23), is used in conjunction with equations (14) and (15), or equations (18) and (19), to calculate ϕ_1 and ϕ_1'.

Solutions

The foregoing equations were solved numerically on a Burroughs 220. Calculations for $Pr = 10^{-2}$, 0·72, 10^2 and 10^4 for the isothermal case and for $Pr = 10^2$ for the uniform-flux case were based upon the simpler approximate relations for ϕ_1, equations (22) and (23) (i.e. f_1 was taken to be zero). This procedure was checked for accuracy by calculating the full ϕ_1, f_1 solution for the isothermal case for $Pr = 10^2$ from (16) and (17). The difference in $\phi_1'(0)$, which is the critical quantity for heat transfer, is 5 %.

The nature of the effect of viscous dissipation upon the temperature distribution within the boundary layer may be seen in figure 1, where ϕ_0 and ϕ_1 are plotted for both the isothermal and uniform-flux cases. The effect is seen to extend throughout the thermal boundary layer and to be a maximum at the wall for the case of uniform flux. The effect upon the heat flux is seen in figure 2. The effect of viscous dissipation is again present throughout the boundary layer, the effect being the greatest at the wall for the isothermal case.

The principal results of the calculations are listed in table 1. For the isothermal case the slope of the temperature perturbation function at $\eta = 0$ is given and for the uniform-flux case the value of the function itself is given at $\eta = 0$. Similar information is also given for the zero-dissipation solutions to a higher accuracy than heretofore available in the literature. In addition, $\phi_0'(0)$ is given for a Prandtl number of 10^4. This value has not previously been available.

It is interesting to note that the ratio $\phi_1'(0)/\phi_0'(0)$ for the isothermal case has closely approached an asymptotic value in the Prandtl-number interval 10^2 to 10^4. However, it was not possible to estimate this value from these results.

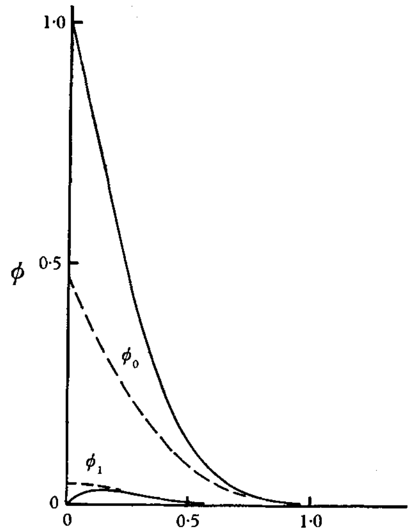

FIGURE 1. Temperature distributions in the boundary layer; Pr = 100; ——, isothermal; – – – uniform flux.

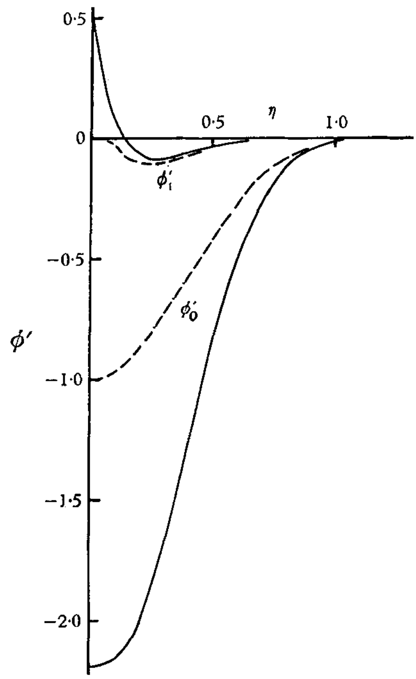

FIGURE 2. Heat-flux distributions in the boundary layer; Pr = 100; ——, isothermal; – – – uniform flux.

	Isothermal			Uniform flux		
Pr	$\phi_0'(0)$	$\phi_1'(0)$	$\phi_1'(0)/\phi_0'(0)$	$\phi_0(0)$	$\phi_1(0)$	$\phi_1(0)/\phi_0(0)$
0·01	−0·080592	0·003497	−0·04340	—	—	—
0·72	−0·50463	0·07506	−0·1487	—	—	—
10^2	−2·1914	0·4877	−0·2226	0·46568	0·04446	0·09547
10^4	−7·0913	1·686	−0·2378	—	—	—

TABLE 1. Results of numerical solutions

Effects upon heat transfer

The effect of viscous dissipation is to inhibit heat transfer (or to require a higher temperature difference) when the surface transfers heat to the fluid and to augment transfer (or to require a lower temperature difference) when heat is transferred to the surface.

For the isothermal case, the local Nusselt number is defined below and is calculated from equation (9) in terms of the solutions of the differential equations as

$$Nu_x = \frac{q_x''}{(t_0-t_\infty)}\frac{x}{k} = -\left(\frac{Gr_x}{4}\right)^{\frac{1}{4}}\phi'(0) = -\phi_0'(0)\left(\frac{Gr_x}{4}\right)^{\frac{1}{4}}\left[1 \pm 4\left(\frac{g\beta x}{c_p}\right)\frac{\phi_1'(0)}{\phi_0'(0)} \pm \ldots\right]. \tag{24}$$

The effect of viscous dissipation on heat transfer is seen to be zero at the leading edge and to increase along the surface. The Nusselt number based upon the average convection coefficient for a surface of length L is

$$Nu_L = \frac{hL}{k} = -\tfrac{4}{3}\phi_0'(0)\left(\frac{Gr_L}{4}\right)^{\frac{1}{4}}\left[1 \pm \frac{12}{7}\left(\frac{g\beta L}{c_p}\right)\frac{\phi_1'(0)}{\phi_0'(0)} \pm \ldots\right]. \tag{25}$$

The limits of the applicability of laminar boundary-layer analysis for natural convection are not known in a general way. Measurements made in air and scattered heat-transfer data suggest that the limit may be expressed in terms of Gr_x and that the range of laminar boundary-layer transport for gases and for a number of common liquids is not less than†

$$10^4 < Gr_x < 5\times 10^8.$$

Viscous dissipation effects, if important, would appear near the upper limit. The upper limit of plate height and temperature difference is estimated as follows:

$$x^3(t_0 - t_\infty) \leqslant 5\times 10^8(\nu^2/g\beta).$$

The effect of viscous dissipation in the case of an assigned surface heat flux is to make necessary a larger difference between t_0 and t_∞ for the convection of a heat flux to the fluid. The surface temperature is calculated from the solution

$$(t_0 - t_\infty) = \frac{(q''x/k)}{(Gr_x/5)^{\frac{1}{5}}}\phi_0(0) = \phi_0(0)\frac{(q''x/k)}{(Gr_x/5)^{\frac{1}{5}}}\left[1 \pm 5\frac{g\beta x}{c_p}\frac{\phi_1(0)}{\phi_0(0)} \pm \ldots\right]. \tag{26}$$

There is an even greater uncertainty concerning the limit of laminar transport on a surface with an assigned flux condition. However, experimental evidence for gases indicates that the limit is in terms of the Grashof number and at a value not greatly different from that which applies for an isothermal surface when the uniform-flux Grashof number is expressed in the same form. The order of magnitude relation between the two Grashof numbers is

$$\frac{g\beta x^4 q''}{k\nu^2} = O\left[\frac{g\beta x^3(t_0 - t_\infty)}{\nu^2}\right]^{\frac{5}{4}}.$$

The limitation on plate height and surface flux is

$$x^4 q'' \leqslant 10^{11}(k\nu^2/g\beta).$$

Discussion

The dissipation number $g\beta x/c_p$ is small for most ordinary engineering devices with common fluids for the gravitational field strength of the earth. For example, the quantity $g\beta/c_p$ remains in the range 10^{-7} to 10^{-4} ft.$^{-1}$ for fluids as different as liquid sodium, mercury, gases at ordinary temperatures, water, and viscous silicones.‡ Therefore, for such fluids, important viscous dissipation would result only at very large values of x not commonly encountered and for which the flow would probably be turbulent.

However, in rotating systems the dissipation number may be large. This may

† See, for example, the discussion in Gebhart (1961).

‡ One important exception to this conclusion is gases (and liquids) at very low temperature, where β/c_p may be large.

occur for flows well within the laminar limit because, although the Grashof number is linearly dependent upon the field strength, it is dependent upon the third power of x. Therefore, the product gx in the dissipation number is proportional to $g^{\frac{2}{3}}$ for a given value of the Grashof number.

The foregoing analysis would apply to natural convection on the radial surfaces of cavities which have a large width compared to the thickness of the convection layer. A number of studies† indicate that this condition is effectively met even for relatively high-aspect-ratio cavities. However, these results and those cited in the Introduction would apply only under conditions in which the Coriolis force arising from the induced flow is small compared to the centrifugal force which induces the flow. These conditions have not been established.

It is necessary to consider whether or not appreciable viscous dissipation necessarily implies processes for which flow-induced pressure gradients are also appreciable. Viscous dissipation is accounted for by the last term in (3) whose magnitude relative to the other terms in the energy equation is indicated by the dissipation number $g\beta x/c_p$. This number is not expressible exclusively in terms of the Grashof and Prandtl numbers. It is shown below that the effect of the induced pressure gradient upon the flow does depend exclusively upon the Grashof and Prandtl numbers and is not, therefore, directly connected to the magnitude of the viscous dissipation effect.

Equation (1) neglects the x-direction pressure gradient which arises in the remote fluid due to the induced flow. The relative effect of the pressure gradient is evaluated by comparing the resulting pressure force with the viscous force in the boundary layer. The pressure gradient, exclusive of the hydrostatic contribution, is calculated from the Bernoulli equation, written outside the boundary layer as

$$p+\tfrac{1}{2}\rho V^2 = \text{const.},$$

where $V = v_\infty$ is the asymptotic value of the velocity component normal to the surface, which arises due to convection. This component is estimated from the boundary-layer solution for the case of an isothermal plate as

$$v_\infty = -\frac{3\nu}{4x^2}\left(\frac{Gr_x}{4}\right)^{\frac{1}{4}} f_0(\infty),$$

where $f_0(\infty)$ is the limit of f_0 as $\eta \to \infty$. The pressure gradient is, therefore,

$$\left(\frac{\partial p}{\partial x}\right) = -\rho v_\infty \frac{\partial v_\infty}{\partial x} = -\frac{9\rho\nu^2}{4x^3}\left(\frac{Gr_x}{4}\right)^{\frac{1}{2}} f_0^2(\infty).$$

The viscous force is converted to similarity variables as follows:

$$\mu\frac{\partial^2 u}{\partial y^2} = \mu\psi_{yyy} = \frac{4\rho\nu^2}{x^3}\left(\frac{Gr_x}{4}\right) f_0'''.$$

Since the pressure gradient normal to the surface is of negligible magnitude for a vertical surface, the pressure gradient calculated above may be applied over the whole boundary layer. Therefore, the ratio of pressure to viscous forces is

$$r = -\frac{9}{8(Gr_x)^{\frac{1}{2}}}\frac{f_0^2(\infty)}{f_0'''} \quad \text{or} \quad r(Gr_x)^{\frac{1}{2}} = -\frac{9}{8}\frac{f_0^2(\infty)}{f_0'''}.$$

† Lighthill (1953), Batchelor (1954), and Poots (1958).

This estimate does not apply near the leading edge because of the singularity at $x = 0$. The quantity $r(Gr_x)^{\frac{1}{4}}$ is thus seen to be a function only of the Prandtl number and varies monotonically from 20 to 0·02 over the Prandtl-number range from 10^{-2} to 10^{2}. Comparisons of the pressure force with the inertia and buoyancy forces give similar results. Thus for $Gr_x > 10^4$, pressure forces may be significant for liquid metals but are negligible for higher Prandtl number fluids.

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